Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MTH101AR Quiz 1 A
February 1, 2013

Roll No: ……………………………………… **Time: 30 Min**

Marks: 15

Name: ……………………………………………….

1. For the sequence $\{x_n\}$, where $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$, show that (i) $\lim_{n\to\infty} x_n$ exists (ii) $\lim_{n\to\infty} x_n$ lies between 1/2 and 1. (5)

   Solution: (i) $\{x_n\}$ is an increasing sequence, since

   $$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} + \frac{1}{n+n+1} = x_{n+1}$$ (1 mark)

   $\{x_n\}$ is bounded above, since $x_n < \frac{n}{n+1} < 1$, for all $n = 1, 2, \ldots \Rightarrow \lim_{n\to\infty} x_n$ exists (2 marks)

   (ii) $\frac{n}{n+n} \le x_n \le \frac{n}{n+1} \Rightarrow \frac{1}{2} \le \lim_{n\to\infty} x_n \le 1$ (2 marks)

2. Let a function $f : [a, b] \to R$ be continuous and $c \in (a, b)$. if $f(c) > 0$ then prove that, for some $\delta > 0$, $f(x) > 0$ in interval $(c-\delta, c+\delta)$ contained in (a, b). (5)

   Solution: f continuous at $c \Rightarrow f(c) - \varepsilon < f(x) < f(c) + \varepsilon$, for x in some small interval $(c-\delta, c+\delta)$ contained in (a, b), some $\delta > 0$. Since $f(c) > 0$, choose $0 < \varepsilon < f(c)$ so that the left hand side inequality gives $f(x) > 0$ in $(c-\delta, c+\delta)$. (0, 3 or 5 marks)

   Alternatively, let $\delta > 0$ does not exist such that $f(x) > 0$ in interval $(c-\delta, c+\delta)$. Then, in every interval $\left(c - \frac{1}{n}, c + \frac{1}{n}\right), n > N,$, contained in (a, b), there exists an x_n, such that $f(x_n) \le 0 \Rightarrow \lim_{n\to\infty} f(x_n) \le 0 \Rightarrow f(c) \le 0$, a contradiction. (0, 3 or 5 marks)

3. For the function $h(x) = x e^{-x^2}$, determine (i) the points of maxima (ii) points of minima (iii) the points of inflection.

   Solution : $h'(x) = e^{-x^2}(1-2x^2),\ h''(x) = e^{-x^2}(4x^3 - 6x)$

   $\Rightarrow x = \frac{1}{\sqrt{2}}$ is point of maxima, $x = \frac{-1}{\sqrt{2}}$ is point of minima , since $h'(\pm\frac{1}{\sqrt{2}}) = 0,\ h''(\frac{1}{\sqrt{2}}) < 0, h''(-\frac{1}{\sqrt{2}}) > 0$

   (1 marks for each correct)

   $x = 0, \pm\sqrt{\frac{3}{2}}$ are points of inflection, since $h''(0) = h''(\pm\sqrt{\frac{3}{2}}) = 0, h''(-\in) > 0, h''(\in) < 0, h''(\pm\sqrt{\frac{3}{2}} - \in) < 0$

   and $h''(\pm\sqrt{\frac{3}{2}} + \in) > 0$ for sufficiently small $\in > 0$. (1 mark for each correct)

   No marks if an answer is not supported with correct justification.

Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MTH101AR Quiz 1 B
February 1, 2013

Roll No: ………………………………………. **Time: 30 Min**
Marks: 15

Name: …………………………………………………..

1. For the sequence $\{x_n\}$, where $x_n = \frac{1}{n+2} + \frac{1}{n+4} + \ldots + \frac{1}{3n}$, show that
(i) $\lim_{n\to\infty} x_n$ exists (ii) $\lim_{n\to\infty} x_n$ lies between 1/3 and 1. (5)
Solution: (i) $\{x_n\}$ is an increasing sequence, since
$$x_n = \frac{1}{n+2} + \frac{1}{n+4} + \ldots + \frac{1}{n+2n} < \frac{1}{n+2} + \frac{1}{n+4} + \ldots + \frac{1}{n+2n} + \frac{1}{3n+3} = x_{n+1}$$
$\{x_n\}$ is bounded above, since $x_n < \frac{n}{n+2} < 1$, for all $n = 1, 2, \ldots \Rightarrow \lim_{n\to\infty} x_n$ exists
(ii) $\frac{n}{n+2n} \le x_n \le \frac{n}{n+2} \Rightarrow \frac{1}{3} \le \lim_{n\to\infty} x_n \le 1$

2. Let a function $f:[a, b] \to R$ be continuous and $c \in (a, b)$. if $f(c) < 0$ then prove that, for some $\delta > 0$, $f(x) < 0$ in interval $(c-\delta, c+\delta)$ contained in (a, b). (5)

Solution: f continuous at $c \Rightarrow f(c) - \varepsilon < f(x) < f(c) + \varepsilon$, for x in some small interval $(c-\delta, c+\delta)$ contained in (a, b), some $\delta > 0$. Since $f(c) < 0$, choose $\varepsilon < -f(c)$ so that the right hand side inequality gives $f(x) < 0$ in $(c-\delta, c+\delta)$).

Alternatively, let $\delta > 0$ does not exist such that $f(x) < 0$ in interval $(c-\delta, c+\delta)$. Then, in every interval $\left(c - \frac{1}{n}, c + \frac{1}{n}\right), n > N,$ contained in (a, b), there exists an x_n, such that $f(x_n) \ge 0 \Rightarrow \lim_{n\to\infty} f(x_n) \ge 0 \Rightarrow f(c) \ge 0$, a contradiction.

3. For the function $h(x) = x e^{-2x^2}, -\infty < x < \infty$, determine (i) the points of maxima (ii) points of minima (iii) the points of inflection.

Solution : $h'(x) = e^{-2x^2}(1-4x^2),\ h''(x) = e^{-2x^2}(16x^3 - 12x)$
$\Rightarrow x = \frac{1}{2}$ is point of maxima, $x = -\frac{1}{2}$ is point of minima, since $h'(\pm\frac{1}{2}) = 0,\ h''(\frac{1}{2}) < 0, h''(-\frac{1}{2}) > 0$.
(1 mark for each correct)

$x = 0, \pm\frac{\sqrt{3}}{2}$ are points of inflection, since $h''(0) = h''(\pm\frac{\sqrt{3}}{2}) = 0$, $h''(-\in) > 0, h''(\in) < 0, h''(\pm\frac{\sqrt{3}}{2} - \in) < 0$ and $h''(\pm\frac{\sqrt{3}}{2} + \in) > 0$ for sufficiently small $\in > 0$. (1 mark for each correct)

No marks if an answer is not supported with correct justification.

Graph of $h(x) = xe^{-x^2}$: